$$\nu(\xi^{(k)}) \leqslant C_0\mu^{-1} \leqslant \delta, \quad \nu(y_1^{(k)}) \leqslant C_1\mu^{-2} \leqslant \delta$$
$$\nu(y_2^{(k)}) \leqslant C_2\mu^{-1} \leqslant \delta \quad (k = 0, 1, \ldots) \tag{8.3}$$

hold for $\mu > \mu_3$. The proof of this assertion is analogous to that of inequalities (4.3) and is carried out using the estimates (7.7), (7.8) and estimates (6.10) in the case when $\eta = 0,\ u = u^{\cdot} = 0$. We introduce the notation

$$a_k = \nu(\xi^{(k)} - \xi^{(k-1)}), \quad b_k = \nu(y_1^{(k)} - y_1^{(k-1)}), \quad c_k = \nu(y_2^{(k)} - y_2^{(k-1)})$$

On the strength of inequalities (6.10), (7.7) and (8.3), when $\mu > \mu_3$ we have

$$a_{k+1} \leqslant \varkappa\left(\frac{a_k}{\mu} + b_k + c_k\right), \quad b_{k+1} \leqslant \frac{\varkappa}{\mu}\left(\frac{a_k + c_k}{\mu} + b_k\right) \tag{8.4}$$
$$c_{k+1} \leqslant \varkappa\left(\frac{a_k + c_k}{\mu} + b_k\right) \quad (k = 1, 2, \ldots)$$
$$\varkappa = K[1 + 2(C_0 + C_1 + C_2)]\max(N_0,\ 2N)$$

Consider the number sequence $\rho_k = a_k\mu^{-1/2} + b_k + c_k\mu^{-1}$ $(k = 1, 2, \ldots)$. As a consequence of (8.4), $\rho_{k+1} \leqslant 3\varkappa\mu^{-1/2}\rho_k$ $(k = 1, 2, \ldots)$. We set $M = \max(\mu_3,\ 36\varkappa^2)$. Then when $\mu > M$ the estimate $\rho_{k+1} \leqslant \rho_k/2$ $(k = 1, 2, \ldots)$ is valid. Using this estimate it can be proved that the sequences $\{\xi^{(k)}(t)\}_{k=0}^{\infty}$, $\{y_j^{(k)}(t)\}_{k=0}^{\infty}$ $(j = 1, 2)$ converge uniformly in the interval $[0, 2\pi]$ to certain continuous functions $\xi^*(t)$, $y_j^*(t)$. The relations

$$\xi^*(0) = \xi^*(2\pi), \quad y_j^*(0) = y_j^*(2\pi) \quad (j = 1, 2)$$
$$\nu(\xi^*) \leqslant C_0\mu^{-1}, \quad \nu(y_1^*) \leqslant C_1\mu^{-2}, \quad \nu(y_2^*) \leqslant C_2\mu^{-1}$$

hold. Passing to the limit in (8.2) as $k \to \infty$, we find that $\xi^*(t)$, $y_1^*(t)$ and $y_2^*(t)$ is the solution of system (8.1). The function $\xi^*(t)$ is continuously differentiable, the function $y_1^*(t)$ is twice continuously differentiable, and $dy_1^*(t)/dt = y_2^*(t)$. Exactly as in the proof of Theorem 1 we can establish that the solution found is unique. Having continued the functions ξ^*, y_1^* 2π-periodically along the whole real axis, we obtain the desired periodic solution of system (6.9).

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THE HAMILTON-JACOBI EQUATION IN DOMAINS OF POSSIBLE MOTIONS WITH A BOUNDARY*

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The problem of the existence of solutions of the truncated Hamilton-Jacobi equation in the whole domain of possible motions with a boundary is investigated. Constraints on the topology of the domains of possible motions, in which the Hamilton-Jacobi equation is solvable in the large, are pointed out. In particular, the boundary cannot be connected. The existence of solutions in the whole domain of possible motions is obstructed by focal points at which infinitely close trajectories leaving the boundary intersect. A connection between the complete integral of the Hamilton-Jacobi equation and the particular solutions in the neighbourhood of the boundary is indicated.

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A smooth n-dimensional manifold M serves as the configuration space of a natural mechanical system with n degrees of freedom, while the tangent bundle TM of the manifold M serves as the phase space. The kinetic energy $T: TM \to R$ is a smooth function in phase space, quadratic with respect to the velocities, while the potential energy $U: M \to R$ is a smooth function on M. The motions are smooth mappings $m: R \to M$ satisfying in local coordinates $q = \{q_i\}$ $(i = 1, \ldots, n)$ on M Lagrange equations with Lagrangian $L = T - U$. For a fixed value of h the integral of the energy $T + U = h$ delineates in the phase space a $(2n - 1)$-dimensional hyperplane Π^{2n-1} on which the system's phase trajectories wholly lie. The natural projection of the hyperplane Π^{2n-1} onto the manifold M defines a domain

$$D = \{U \leqslant h\} \subset M \tag{0.1}$$

in which a motion can take place, i.e., a domain of possible motions.

According to the principle of least action, inside D the system's trajectories coincide with the geodesic lines of the Jacobi metric $d\rho^2 = (h - U)\, ds^2$, where ds^2 is the Riemann metric on M specifying the kinetic energy, i.e., $T = {}^1/_2\, (ds/dt)^2$. Inside domain D the metric $d\rho$ is the usual Riemann metric, while on the boundary $\partial D = \{U = h\}$ it has a singularity: $d\rho = 0$, i.e., the length of any curve on the boundary equals zero. The geometry of the geodesics in domains with a boundary is similar to the usual geometry of Riemann spaces /1/.

1. Distance from the boundary as a function of "truncated action".

Everywhere below we assume that there are no equilibrium positions -- critical points of potential energy U -- on the boundary ∂D. Then ∂D is a smooth $(n - 1)$-dimensional manifold. The distance of a point $q \in D$ up to the boundary ∂D is, by definition, the quantity

$$\partial (q) = \inf_{a \in \partial D} d\,(q, a)$$

where $d\,(a, b)$ denotes the lower bound on lengths in the metric $d\rho$ of piecewise-smooth curves from domain D with ends at points a and b. It has been proved /1/ that for an arbitrary point q of a compact domain D a trajectory exists, passing through it, which reaches the boundary and whose length equals $\partial\,(q)$. Hence it follows, in particular, that the set of all trajectories starting at boundary points fills up the whole set D.

We denote by Σ_p the set of points from D whose distance up to the boundary ∂D along trajectories starting from the boundary equals p.

Lemma 1.1. A $p_0 > 0$ exists (p_0 small) such that the distance $\partial\,(q) = p$ for all points $q \in \Sigma_p$, $0 \leqslant p \leqslant p_0$. The set Σ_p is a smooth hypersurface in D, diffeomorphic with ∂D.

Lemma 1.2 (the analog of Gauss' lemma). A $p_0 > 0$ exists such that for all $p \in [0, p_0]$ the geodesics starting from the points of the boundary ∂D intersect the hypersurfaces Σ_p at right angles.

Lemmas 1.1 and 1.2 are proved in /1,2/. From Gauss' lemma it follows that a neighbourhood of the boundary ∂D exists, i.e., a strip contained between ∂D and Σ_p (p is fairly small), in which the trajectories starting from boundary points do not intersect. Consequently, a single trajectory of this family passes through each point of the small neighbourhood. Hence, in a small neighbourhood of ∂D the integral of truncated action

$$S = \int_\gamma \sqrt{h - U}\, ds = \int_\gamma \sqrt{h - U}\, \sqrt{T}\, dt = \int_\gamma 2T\, dt$$

computed along a trajectory leaving the boundary, can be treated as a function $S\,(q)$ of a finite point q, which, obviously, equals the distance $\partial\,(q)$.

Statement 1.1. The differential of the function S equals

$$dS = p\,dq = \sum_{i=1}^{n} p_i\, dq_i$$

where $p = \partial T/\partial q^{\cdot}$ is determined with respect to the velocity $q^{\cdot}$ on the trajectory γ.

This statement is analogous to the theorem on the differential of the action function (/3/, Chapter 9) and can be proved by the same method. From the energy integral and the preceding statement we have the following:

Statement 1.2. The function $S\,(q)$ satisfies the equation

$$H\,(\partial S/\partial q,\ q) = h$$

called the truncated Hamilton-Jacobi equation.

Since $H = T + U$, where $T = {}^1/_2\, \langle A\,(q)p,\ p\rangle$ is a positive-definite quadratic form ($\langle , \rangle$ is the scalar product), the Hamilton-Jacobi equation can be written as

$${}^1/_2\, \langle A\,(q)\, \partial S/\partial q,\ \partial S/\partial q\rangle + U\,(q) = h \tag{1.1}$$

We will investigate the solution of Eq.(1.1) in the domain D. Since $dS = 0$ on the boundary, we mean by a solution a smooth function inside the domain D, which takes constant values on each connected component of the boundary ∂D.

2. The necessary conditions for the Hamilton-Jacobi equation to be solvable in the large. *Theorem 2.1.* A smooth solution of Eq.(1.1) does not exist in the whole domain of possible motions with connected boundary ∂D.

Proof. Assuming a solution S to exist, we conclude that it attains extremal values inside the domain D. Since the gradient of the function S equals zero at an extremal point, it follows from (1.1) that this point must be on the boundary ∂D. We thus have a contradiction.

On the question of whether domains D are possible in which a solution of Eq.(1.1) in the large exists, we have the following:

Theorem 2.2. If a solution of Eq.(1.1) exists in the whole domain D is diffeomorphic with the direct product

$$\Gamma \times [0, 1](\partial D = \Gamma \times \{0\} \cup \Gamma \times \{1\})$$

where Γ is a smooth connected compact $(n-1)$ dimensional manifold.

Proof. We assume the existence of a solution S of Eq.(1.1) in the whole domain (0.1) of possible motions. We have already shown that the boundary ∂D of domain D cannot be connected. First we consider the case when ∂D is divided into two non-intersecting smooth connected manifolds: $\partial D'$ and $\partial D''$ $(\partial D = \partial D' \cup \partial D'', \partial D' \cap \partial D'' = \varnothing)$ and $S|_{\partial D'} = c_1 = \text{const}, S|_{\partial D''} = c_2 = \text{const}$. We denote by Σ_ε the set of points of D a distance $\varepsilon > 0$ in the Jacobi metric from the boundary. Let Σ_ε' and Σ_ε'' be smooth non-intersecting manifolds comprising Σ_ε, i.e., $\Sigma_\varepsilon = \Sigma_\varepsilon' \cup \Sigma_\varepsilon''$ $(\Sigma_\varepsilon' \cap \Sigma_\varepsilon'' = \varnothing)$, and Σ_ε' (Σ_ε'') is close to $\partial D'$ $(\partial D'')$. The manifold Σ_ε bounds some smooth manifold with boundary G, which is contained in D. The function S takes constant values on Σ_ε' and Σ_ε'' and has no critical points on them. Therefore, S can be treated as the Morse function of the triad $(G, \Sigma_\varepsilon', \Sigma_\varepsilon'')$ of smooth manifolds. Since S does not have critical points in G, the triad's Morse number μ equals zero. From the trivial cobordism theorem /4/ it follows that $(G, \Sigma_\varepsilon', \Sigma_\varepsilon'')$ is a trivial cobordism, i.e., G is diffeomorphic with $\Sigma_\varepsilon' \times [0, 1]$ and Σ_ε'' is diffeomorphic with Σ_ε'. On the other hand, $\partial D'$ and Σ_ε' $(\partial D''$ and $\Sigma_\varepsilon'')$ also bound a smooth manifold with boundary, diffeomorphic with $\partial D' \times [0, 1]$ $(\partial D'' \times [0, 1])$. This has been shown, for instance, in /1/. By joining the manifolds together along common boundaries, i.e., by identifying $\partial D' \times \{1\}$ and Σ_ε' $(\Sigma_\varepsilon' \times \{1\}$ and $\partial D'' \times \{1\})$, we obtain that D is diffeomorphic with $\partial D' \times [0, 1]$. The assumption that ∂D is divided into only two connectivity components does not reduce the proof's generality. Indeed, by assuming the existence of more than two connectivity components we conclude that D is partitioned into several manifolds with a boundary, each of which is diffeomorphic with a direct product of one boundary component and an interval. The theorem is proved.

Corollary. If a solution of the Hamilton-Jacobi equation exists in the whole domain of possible motions, then each trajectory starting from the boundary corresponds to a libration motion.

Proof. We take some trajectory starting from the boundary. Differentiating S along the trajectory, we get that inside D

$$\frac{dS}{dt} = \frac{\partial S}{\partial q} q^{\cdot} = 2(h - U) > 0$$

Consequently, S increases along this trajectory, and if we assume that $S(q(t))|_{t=0} = c_1$, then at a specific t' the function S attains a value c_2 and the moving point is incident on the boundary. The number t' is finite since a point cannot asymptotically approach ∂D as $t \to \infty$ /1/. The assertion is proved.

Example. Let $M = R^2\{x, y\}$, $2T = x^{\cdot 2} + y^{\cdot 2}$, $2U = x^2 + y^2 - 2a\sqrt{x^2 + y^2} + a^2$, $a > 0$. The domain of possible motions is the ring

$$(a - \sqrt{2h})^2 \leqslant x^2 + y^2 \leqslant (a + \sqrt{2h})^2, \quad h < a^2/2$$

The function

$$S(x, y) = \frac{1}{2}\left[(\sqrt{2U} - a)\sqrt{2h - (\sqrt{2U} - a)^2} + 2h \arcsin \frac{\sqrt{2U} - a}{\sqrt{2h}}\right]$$

satisfies the Hamilton-Jacobi equation, is smooth inside D and takes constant values on the boundary of the domain D.

3. Focal points of the boundary of the domain of possible motions. Assume that the portrait with local coordinates $q = \{q_i\}$ $(i = 1, \ldots, n)$ covers the whole domain D. On the boundary ∂D, in some neighbourhood of an arbitrary point q_0 we take the local coordinates $\alpha = \{\alpha_i\}(i = 1, \ldots, n-1)$ such that $\alpha = 0$ corresponds to point q_0. We consider the solutions of the equations of motion with the initial conditions

$$q = q(t, \alpha), \quad q(0) = q_0, \; q^{\cdot}(0) = 0 \tag{3.1}$$

as an $(n-1)$-parametric family of trajectories γ_α starting from a neighbourhood of point q_0 of the boundary ∂D. Thus, we have defined the mapping

$$F : \partial D \times R_+ \to D, \quad R_+ = \{t,\ t > 0\}$$

specified by the formula $F(t, \alpha) = q(t, \alpha)$.

Definition. A point q^* is called a focal point of the boundary ∂D along the trajectory γ_0 if it is a critical value of the mapping F.

In other words, if $q^* = q(t^*, 0)$ is a focal point, then the Jacobian of the mapping F at the point $(t^*, 0)$ is singular, i.e.,

$$\operatorname{rank} \| \partial q / \partial \beta \|_{(t^*, 0)} < n, \quad \beta = (\alpha, t) \tag{3.2}$$

The next theorem clarifies the geometrical meaning of this definition.

Theorem 3.1. If q^* is a focal point, then either $q^* \in \partial D$ or the trajectories starting from infinitely close boundary points intersect at the point q^*.

Using Lemmas 1.1 and 1.2 we derive the following:

Corollary. A neighbourhood of the boundary ∂D exists in which there are no focal points.

Proof of the theorem. Together with the trajectory $q(t, 0)$ we consider an infinitely close trajectory $q + \delta q$, where $\delta q = \langle (\partial q / \partial \alpha)|_{\alpha=0},\ \delta\alpha \rangle$. At the intersection point $(q + \delta q)|_{t^*+\varepsilon} = q(t^*)$. Since

$$(q + \delta q)|_{t^*+\varepsilon} = q(t^*) + q^{\cdot}(t^*)\,\varepsilon + \delta q|_{t^*} + \ldots$$

we obtain, by dropping terms of a higher order of smallness, the equality

$$q^{\cdot}(t^*)\,\varepsilon + \delta q|_{t^*} = 0$$

or

$$q^{\cdot}(t^*)\,\varepsilon + \langle (\partial q / \partial \alpha)|_{t=t^*,\ \alpha=0},\ \delta\alpha \rangle = 0 \tag{3.3}$$

The resultant system has a non-trivial solution if condition (3.2) is satisfied. If $q^{\cdot}(t^*) \neq 0$, then q^* — the intersection point — lies inside the domain D. When $q^{\cdot}(t^*) = 0$, an intersection of the trajectories does not always follow from (3.3). In this case $q^* \in \partial D$.

Theorem 3.2. If an envelope of the family γ_α exists, then the focal point of the boundary along γ_0 coincides with the common point of the envelope and the trajectory γ_0.

Proof. Assume that an envelope prescribed by the equation $f(q) = 0$ exists, which defines a regular hypersurface in D. Let the point of tangency of the trajectory γ_α and the envelope correspond to the instant $t^*(\alpha)$. Then in some neighbourhood on ∂D of the point q_0 we shall have $f(q(t^*(\alpha), \alpha)) \equiv 0$. Differentiating this identity with respect to α and keeping in mind the tangency condition

$$\left\langle \frac{\partial f}{\partial q}\Big|_{q^*},\ \frac{\partial q}{\partial t}\Big|_{t^*} \right\rangle = 0 \tag{3.4}$$

we obtain

$$\left\langle \frac{\partial f}{\partial q}\Big|_{q^*},\ \frac{\partial q}{\partial \alpha}\Big|_{t=t^*,\ \alpha=0} \right\rangle = 0 \tag{3.5}$$

From (3.4), (3.5) and the regularity condition for the envelope it follows that

$$\det \| \partial q / \partial \beta \|_{(t^*, 0)} = 0 \quad ((t, \alpha) = \beta)$$

which proves the theorem. Consider the hypersurface Σ_ε defined by Lemma 1.1. The trajectories starting from the boundary ∂D intersect the hypersurface Σ_ε at right angles. Since inside the domain D each trajectory is a smooth curve, on it we can introduce a natural parameter and it becomes a "standard" geodesic. From an arbitrary point $m \in \Sigma_\varepsilon$ we draw a geodesic in the direction of the normal, which we write as

$$\gamma(k) = \exp_m(kn); \quad \gamma(0) = m,\ \gamma^{\cdot}(0) = n$$

where n is a vector perpendicular to $T_m \Sigma_\varepsilon$. Thus we have defined a mapping $\exp: \perp \Sigma_\varepsilon \to D$ of the normal bundle of the hypersurface Σ_ε into D. Since the geodesics obtained coincide with the trajectories starting from boundary ∂D (two geodesics tangent to each other at some point coincide), the critical values of the mapping F and of the mapping exp coincide inside domain D. A critical value of the exponential mapping of the normal bundle of the submanifold $N \subset M$ in M is, by definition, called a focal point of the submanifold N /5/. Thus, we have proved the following:

Lemma 3.1. A focal point of the boundary ∂D inside domain D coincides with a focal point of the hypersurface Σ_ε, where ε is fairly small.

Theorem 3.3. (the analog of Jacobi's theorem). The distance up to ∂D is not minimized after the first focal point of a trajectory starting from the boundary ∂D.

Proof. Since the focal points of the boundary ∂D and the hypersurface Σ_ε coincide inside $D, \gamma(t)$ does not minimize the distance up to Σ_ε (/5/, Chapter 11). Since the distance of each point $m \in \Sigma_\varepsilon$ up to ∂D equals ε /1/, from this result the theorem's assertion follows for internal focal points. If the first focal point $\gamma(t^*)$ lies on the boundary ∂D, this trajectory corresponds to a libration motion. The point $\gamma(t^* + \delta)$ coincides with $\gamma(t^* - \delta)$, therefore, the segment $\gamma([0, t^* - \delta])$ is shorter than the segment $\gamma([0, t^* + \delta])$.

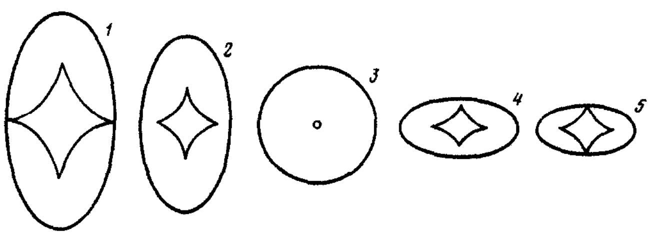
Note. The same assertion is true for a neighbourhood of the point $\gamma(0)$ in ∂D.

If $\gamma([0, t_1])$ is a segment of the trajectory $\gamma(t)$, on which there are no focal points of the boundary ∂D, then, using the statement from /5/ and arguing as above, we conclude that a neighbourhood of the segment in D and a neighbourhood V of the point $\gamma(0)$ in ∂D exist such that γ minimizes the distance among the paths joining the points of V with $\gamma(t)$, $t < t_1$. Thus, the first focal point can be characterized as follows: the point $\gamma(t^*)$ is the first focal point of the boundary ∂D along γ if $\gamma([0, t_1])$ does not minimize the length of the arc up to a neighbourhood of $\gamma(0)$ is ∂D when $t_1 > t^*$, but does minimize it when $t_1 < t^*$. The locus of the focal points of the boundary ∂D, by analogy with geometrical optics, is called a caustic. In the general case it is a manifold of dimension $n - 1$ which can have singularities. As a rule it is very difficult to determine the caustic. We will give two examples not requiring very complicated computations.

Example 1. Let

$$M = R^2\{x, y\},\ 2T = x'^2 + y'^2,\ 2U = x^2 + \omega^2 y^2$$

The domain of possible motions is D: $x^2 + \omega^2 y^2 \leqslant 2h$, where $h > 0$ is the constant total energy. The figure shows the change in the caustic as the parameter ω grows from 0 to ∞ (portraits 1-5 correspond to the values $0 < \omega \leqslant 1/2$, $1/2 < \omega < 1$, $\omega = 1$, $1 < \omega < 2$, $2 \leqslant \omega < \infty$) .



Figure

Example 2. Let

$$M = R^2\{x, y\},\ 2T = x'^2 + y'^2,\ 2U = x^2 - a^2 y^2,\ a = \mathrm{const}$$

The domain is D: $x^2 + a^2 y^2 \leqslant 2h$, where $h < 0$. The caustic has a singularity -- a cusp -- which is located at a distance $\sqrt{-1/_2 h}\,(1 + \operatorname{ch} {}^1/_2 a\pi)$ from the origin. As the constant h decreases, the caustic's spike approaches the origin. We note that the definition of the caustic of the boundary ∂D has been introduced independently of the Hamilton-Jacobi equation and the caustic characterizes the geometry of the geodesics in the domain of possible motions with a boundary.

4. On a solution in a neighbourhood of the boundary of the domain of possible motions.

Without loss of generality we will assume that the boundary of the domain of possible motions is connected. We pose the following problem: it is required to find a solution of Eq.(1.1) which takes a constant value on the boundary ∂D, i.e., $S|_{\partial D} = a = \mathrm{const}$. (The solution of this problem is not unique. Indeed, if $S_1 = S$ is a solution, then $S_2 = 2a - S$ is a solution also).

Theorem 4.1. The functions

$$S(q) = a \pm \int_{q_0 \in \partial D}^{q} 2T\, dt$$

(the integration is along a trajectory connecting the point q with ∂D) are unique solutions of the above problem.

Proof. Following the method of characteristics /6/, we arrive at a process which uniquely leads to the solution $S(q)$, apart from the sign $\pm$. The reason for the ambiguity is the invertibility of the equation of motion.

The theorem asserts that in a small neighbourhood of the boundary we can obtain an integral surface starting from the boundary. Let us consider the possibility of prolonging the solution "along trajectories", i.e., into a neighbourhood of some previously-fixed trajectory starting from the point $q_0 \in \partial D$, which corresponds to the value $\alpha = 0$. In spite of the fact that $q(t, \alpha)$ and $S(t, \alpha)$ are uniquely determined by the system's characteristics, one cannot prolong the integral surface without limit without hitting singular points. Singular points are points at which the function $S(q)$ cannot be single-valued. Non-single-valuedness arises at those points when it is impossible to express t, α uniquely in terms of q. The hypotheses of the inverse function theorem are violated at these points, i.e., the Jacobian vanishes, which reduces to the definition of the caustic. Thus, a part of the caustic serves as a boundary of a single-valued prolongation of the solution of Eq.(1.1). Let us estimate the domain in which a single-valued solution exists. From each point $q_0 \in \partial D$ we issue a trajectory and find the first focal point q_0^*. The distance of the point q_0^* from q_0 along the trajectory equals $\partial(q_0^*)$. We shift ∂D along the trajectory by a distance

$$c = \inf_{q_0 \in \partial D} \partial(q_0^*)$$

We obtain a hypersurface Σ_c, which together with ∂D separates from D a strip in which

$$\det \| \partial q / \partial(t, \alpha) \| \neq 0$$

and a solution of Eq.(1.1) automatically exists. If dq is a displacement along the hypersurface Σ_b $(b < c)$, then

$$0 = dS|_{\Sigma_b} = p\,dq$$

whence it follows that the trajectories intersect the hypersurface at right angles. Thus we have obtained an estimate for the number ρ_0 in the Gauss Lemma 1.2, i.e., ρ_0 must be less than the distance up to the nearest focal point of the boundary ∂D.

In the Hamilton-Jacobi theory an essential role is played by complete integrals, namely, solutions of Eq.(1.1) containing besides h a further $n-1$ constants $a = (a_1, \ldots, a_{n-1})$: $S = S(q, a, h)$ and satisfying the non-singularity condition

$$\det \left\| \frac{\partial^2 S}{\partial \xi\, \partial q} \right\| \neq 0, \quad \xi = (a, h)$$

Let us establish the connection between a complete integral and a solution of Eq.(1.1) in the neighbourhood of the boundary. For simplicity we restrict ourselves to the case of two degrees of freedom. For a fixed value of h the complete integral $S(q, a, h)$ represents a single-parameter family of solutions of Eq.(1.1). The domain D_a of these solutions clearly depend on a and comprises a certain part of the domain of possible solutions. Since for different values of a the functions $S(q, a, h)$ represent different solutions of the Hamilton-Jacobi equation, it follows from Theorem 4.1 that the closure of the domain D_a can intersect the boundary ∂D only along an isolated set of points. Suppose that this condition is satisfied. If an envelope of the family of solutions $S(q, a, h)$ exists, this function too is a solution /6/. Consequently, this envelope is identical with one of the earlier defined solutions $S(q)$ in the neighbourhood of the boundary ∂D.

Example. Let

$$2T = x^{\cdot 2} + y^{\cdot 2},\ 2U = x^2 + y^2,\ h = \text{const};\ D = \{x^2 + y^2 \leqslant 2h\}$$

The Hamilton-Jacobi equation is

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + x^2 + y^2 = 2h$$

Using the method of separation of variables, we find the complete integral S, and taking into account the equation $\partial S / \partial a = 0$ we obtain the envelope.

$$S(x, y) = \frac{1}{2} \sqrt{x^2 + y^2} \sqrt{2h - (x^2 + y^2)} + h \arcsin \frac{\sqrt{x^2 + y^2}}{\sqrt{2h}}$$

A direct verification shows that it indeed represents a solution in the neighbourhood of the boundary.

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